Constructing a supersymmetric generalization of the Gross-Neveu model

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A class of 1+1 dimensional supersymmetric theories with four-fermionic interaction will be built from scratch. The vacua of selected examples will be examined in the 't Hooft limit and compared to the Gross-Neveu model.

INTRODUCTION

In this paper I want to generalize the Gross-Neveu model to a supersymmetric theory. The questions I try to answer are: is there such a theory? And if yes, how does it compare to the standard Gross-Neveu model?

The Gross-Neveu-Model[1] is a renormalizable relativistic field theory in 1+1 dimensions that has a four-fermion interaction ("ψ^4 theory"), shows asymptotic freedom and is analytically soluble in the 't Hooft limit ($N \to \infty$ while keeping Ng^2 constant with N the number of flavours and g^2 the coupling constant of the ψ^4 interaction) even at finite temperature and density ([2] provides an overview).

Supersymmetry (SUSY), the transformation of fermions into bosonic partners ($\delta_\xi \psi \propto \Phi\xi$) and vice versa ($\delta_\xi \Phi \propto \psi\xi$), is in 3+1 dimensions a candidate for physics beyond the standard model. Even (softly) broken it would solve the hierarchy problem in the Higgs sector (why do bare mass and quantum corrections compensate to a physical mass several orders of magnitude smaller?) and provide a possible dark matter particle. While in 3+1 dimensions supersymmetric ψ^4 theories have been looked at (e.g. SNJL in [3]), supersymmetric theories in 1+1 dimensions in general concentrate on interactions terms of the form $\bar{\psi}\psi V(\Phi) + V^2(\Phi)$. A thorough introduction into this field is given by [4] while [5] provides additional information on techniques for low dimensions and the use of Majorana spinors.

This paper follows my diploma thesis([6]). The first is dedicated to the formulation of general supersymmetric theories in 1+1 dimensions. Instead of using the elegant but very formal superfield ansatz I will build the theory from the free theory and check all classes of interaction terms for supersymmetric invariance by hand. In the second part I will select the most Gross-Neveu like theories and examine their vaccuum in the 't Hooft limit.

BUILDING THE SUPERSYMMETRIC THEORY

Basics and free theory

The theory is formulated in the usual way with Majorana spinors and real scalar fields. For the Dirac matrices the following Majorana representation is used

$$\gamma^0 = \sigma_2, \quad \gamma^1 = i\sigma_1, \quad \gamma_5 = -\gamma_0\gamma_1 = \sigma_3.$$

In this representation for Majorana spinors and their bilinears the following relations hold true

$$\begin{aligned}
&\psi^* = \psi, \\
&\bar{\xi}\psi = \bar{\psi}\xi, \quad \bar{\xi}\gamma^\mu\psi = -\bar{\psi}\gamma^\mu\xi, \quad \bar{\xi}\gamma_5\psi = -\bar{\psi}\gamma_5\xi, \\
&2\psi\bar{\xi} = -(\bar{\xi}\psi) - \gamma_\mu(\bar{\xi}\gamma^\mu) - \gamma_5(\bar{\xi}\gamma_5\psi) \quad \text{(Fierz identity)}.
\end{aligned}$$

The Lagrangian of the free theorie with a scalar field Φ, a Majorana field ψ and an auxiliary scalar field F, that is needed to match bosonic and fermionic off-shell degrees of freedom and can be eliminated via the Euler-Lagrange equation, reads

$$\mathcal{L} = (\partial_\mu\Phi)(\partial^\mu\Phi) + i\bar{\psi}\gamma^\mu\partial_\mu\psi + F^2$$

and is invariant under the following SUSY transformations

$$\begin{aligned}
\delta_\xi\Phi &= \bar{\psi}\xi, \\
\delta_\xi\psi &= -i(\partial_\mu\Phi)\gamma^\mu\xi - F\xi, \\
\delta_\xi F &= -i(\partial_\mu\bar{\psi})\gamma^\mu\xi,
\end{aligned}$$

where the parameter ξ is a constant Majorana spinor. These transformations are determined by demanding linearity in parameter and fields, correct Lorentz transformations and correct mass dimensions. Introducing flavours, necessery to obtain ψ^4 interactions and labelled by $a = 1 \dots N$, yields

$$\begin{aligned}
\mathcal{L} &= (\partial_\mu\Phi_a)(\partial^\mu\Phi_a) + i\bar{\psi}_a\gamma^\mu\partial_\mu\psi_a + F_aF_a, \\
\delta_\xi\Phi_a &= \bar{\psi}_a\xi, \\
\delta_\xi\psi_a &= -i(\partial_\mu\Phi_a)\gamma^\mu\xi - F_a\xi, \\
\delta_\xi F_a &= -i(\partial_\mu\bar{\psi}_a)\gamma^\mu\xi.
\end{aligned}$$

Interaction terms

To obtain a theory with interactions I collect all possible combinations of the fields that are Lorentz scalar and invariant under $O(N)$ flavour transformations and sort them by mass dimension. Then I try to find a set of coefficients that yields a SUSY invariant Lagrangian density.

$\mathcal{O}(M)$ interaction terms

The possible field combinations with mass dimension M^1 (and consequently interaction terms with a coupling of dimension M^1) are

$$(a) = \bar{\psi}_a \psi_a (\Phi)^{2l_a},$$
$$(b) = \Phi_a \bar{\psi}_a \psi_b \Phi_b (\Phi)^{2l_b},$$
$$(c) = \Phi_a F_a (\Phi)^{2l_c}.$$

The factor $(\Phi)^{2l} := (\Phi_a \Phi_a)^l$ stems from the fact that every term can be multiplied by arbitary powers of $\Phi_a \Phi_a$ without changing mass dimension or behaviour under Lorentz and flavour transformations. Calculating the SUSY transformations and using the Fierz identity to simplify some of the terms, I obtain

$$\begin{aligned}
\delta_\xi (a) =& 2i(\partial_\mu \Phi_a)\bar{\xi}\gamma^\mu \psi_a (\Phi)^{2l_a} - 2F_a \bar{\xi}\psi_a (\Phi)^{2l_a} + \\
& + 2l_a \bar{\psi}_a \psi_a \bar{\xi}\psi_b \Phi_b (\Phi)^{2l_a - 2}, \\
\delta_\xi (b) =& 2i(\partial_\mu \Phi_a)\Phi_a \Phi_b \bar{\xi}\gamma^\mu \psi_b (\Phi)^{2l_b} + 2\bar{\psi}_a \psi_b \bar{\xi}\psi_a \Phi_b (\Phi)^{2l_b} - \\
& - 2F_a \Phi_a \Phi_b \bar{\xi}\psi_b (\Phi)^{2l_b} + \\
& + 2l_b \bar{\psi}_a \psi_b \bar{\xi}\psi_c \Phi_a \Phi_b \Phi_c (\Phi)^{2l_b - 2} = \\
=& 2i(\partial_\mu \Phi_a)\Phi_a \Phi_b \bar{\xi}\gamma^\mu \psi_b (\Phi)^{2l_b} - \bar{\psi}_a \psi_a \bar{\xi}\psi_b \Phi_b (\Phi)^{2l_b} - \\
& - 2F_a \Phi_a \Phi_b \bar{\xi}\psi_b (\Phi)^{2l_b}, \\
\delta_\xi (c) =& i\Phi_a \bar{\xi}\gamma^\mu (\partial_\mu \psi_a)(\Phi)^{2l_c} + F_a \bar{\xi}\psi_a (\Phi)^{2l_c} + \\
& + 2l_c F_a \Phi_a \Phi_b \bar{\xi}\psi_b (\Phi)^{2l_c - 2}.
\end{aligned}$$

Calculating the interaction Lagrangian density for a fixed number of fields (since supersymmetry does not change the total number of fields) and collecting the same combinations of fields and derivatives yields

$$\begin{aligned}
\delta_\xi \mathcal{L}_1^{(l)} =& \beta_a (a) + \beta_b (b) + \beta_c (c) = \\
=& 2i\beta_a (\partial_\mu \Phi_a)\bar{\xi}\gamma^\mu \psi_a (\Phi)^{2l_a} + \\
& + i\beta_c \Phi_a \bar{\xi}\gamma^\mu (\partial_\mu \psi_a)(\Phi)^{2l_c} + \\
& + 2i\beta_b (\partial_\mu \Phi_a)\Phi_a \Phi_b \bar{\xi}\gamma^\mu \psi_b (\Phi)^{2l_b} + \\
& + \Big(-2\beta_a (\Phi)^{2l_a} + \beta_c (\Phi)^{2l_c} \Big) F_a \bar{\xi}\psi_a + \\
& + \Big(2l_a \beta_a (\Phi)^{2l_a - 2} - \beta_b (\Phi)^{2l_b} \Big) \bar{\psi}_a \psi_a \bar{\xi}\psi_b \Phi_b + \\
& + \Big(-2\beta_b (\Phi)^{2l_b} + 2l_c \beta_c (\Phi)^{2l_c - 2} \Big) F_a \Phi_a \Phi_b \bar{\xi}\psi_b.
\end{aligned}$$

The first three terms can be combined to a total derivative for $\beta_a = \frac{1}{2}\beta_c$, $\beta_b = l_c \beta_c$ and $l_a = l_b + 1 = l_c$. The last three terms vanish with the same relations of $\beta_a, \beta_b, \beta_c$. The complete Lagrangian density of interaction terms of mass dimension M^1 can be written as

$$\mathcal{L}_{int}^M = \big(\tfrac{1}{2}\bar{\psi}_a \psi_a + F_a \Phi_a\big) W_1 + \Phi_a \bar{\psi}_a \psi_b \Phi_b W_1'$$

with

$$W_1(\Phi^2) := \sum_{l=0}^{\infty} m_l (\Phi_a \Phi_a)^l, \qquad W_1'(\Phi^2) := \frac{\partial W_1}{\partial (\Phi^2)}.$$

Combining the free theory with an interaction Lagrangian given by $W_1 = -m$ results, after eliminating the auxiliary field, in the Lagrangian of free massive scalar and Majorana fields with mass m.

$\mathcal{O}(M^2)$ interaction terms

All possible interaction terms with massless coupling (and field combinations with dimension M^2) are

$$\begin{aligned}
(1) &= \bar{\psi}_a \psi_a \bar{\psi}_b \psi_b (\Phi)^{2n_1} \quad \text{(the interesting term)}, \\
(2) &= \bar{\psi}_a \psi_b \bar{\psi}_c \psi_c \Phi_a \Phi_b (\Phi)^{2n_2}, \\
(\bar{1}) &= F_a F_a (\Phi)^{2\bar{n}_1}, \\
(\bar{2}) &= F_a F_b \Phi_a \Phi_b (\Phi)^{2\bar{n}_2}, \\
(\hat{1}) &= F_a \Phi_a \bar{\psi}_b \psi_b (\Phi)^{2\hat{n}_1}, \\
(\hat{2}) &= F_a \Phi_b \bar{\psi}_a \psi_b (\Phi)^{2\hat{n}_2}, \\
(\hat{3}) &= F_a \Phi_a \Phi_b \Phi_c \bar{\psi}_b \psi_c (\Phi)^{2\hat{n}_3}, \\
(A) &= \Phi_a (\partial_\mu \Phi_a)\Phi_b (\partial^\mu \Phi_b)(\Phi)^{2n_A}, \\
(B) &= \Phi_a (\partial_\mu \Phi_b)\Phi_a (\partial^\mu \Phi_b)(\Phi)^{2n_B}, \\
(C) &= (i\bar{\psi}_a \gamma^\mu \partial_\mu \psi_a)\Phi_b \Phi_b (\Phi)^{2n_C}, \\
(D) &= (i\bar{\psi}_a \gamma^\mu \partial_\mu \psi_b)\Phi_a \Phi_b (\Phi)^{2n_D}, \\
(G) &= i\bar{\psi}_a \gamma^\mu \psi_b (\partial_\mu \Phi_a)\Phi_b (\Phi)^{2n_G}.
\end{aligned}$$

Using the same procedure as in the M^1 case (resulting in 29 equations for the relations of 12 coefficients instead of 4 for 3, full calculation in [6, ch. 3.4]) I obtain

$$\begin{aligned}
&\mathcal{L}_{int}^{M^2} = \\
=& \Big((\partial_\mu \Phi_a)(\partial^\mu \Phi_a) + i\bar{\psi}_a \gamma^\mu (\partial_\mu \psi_a) + F_a F_a \Big) W_2 + \\
& + \Phi_a (\partial_\mu \Phi_a)\Phi_b (\partial^\mu \Phi_b) W_2' + i\bar{\psi}_a \gamma^\mu (\partial_\mu \psi_b)\Phi_a \Phi_b W_2' + \\
& + i\bar{\psi}_a \gamma^\mu \psi_b (\partial_\mu \Phi_a)\Phi_b W_2' + \Phi_a F_a F_b \Phi_b W_2' + \\
& + 2F_a \bar{\psi}_a \psi_b \Phi_b W_2' - \tfrac{1}{4}\bar{\psi}_a \psi_a \bar{\psi}_b \psi_b W_2' - \\
& - \tfrac{1}{2}\bar{\psi}_a \psi_a \Phi_b \bar{\psi}_b \psi_c \Phi_c W_2'' + F_a \Phi_a \Phi_b \bar{\psi}_b \psi_c \Phi_c W_2'', \\
& W_2(\Phi^2) := \sum_{n=0}^{\infty} \lambda_n \Phi^{2n}, \quad W_2' := \frac{\partial W_2}{\partial (\Phi^2)}.
\end{aligned}$$

The case $W_2 = \frac{1}{2}$ reproduces the free theory, $W_2 = \frac{1}{2} - g^2 \Phi_b \Phi_b$ yields the most simple supersymmetric theory with ψ^4 interaction.

EXAMINING THE THEORY IN THE 'T HOOFT LIMIT

To obtain the 't Hooft limit I first eliminate the auxiliary field F, then take the usual Euler-Lagrange equations for ψ and Φ, replace all flavour singlets by their vacuum expectation values and use the fact that these vanish for field combinations that are not Lorentz scalar.

The remaining expectation values are

$$\langle\Phi_b\Phi_b\rangle := N\sigma_0, \qquad \langle(\partial_\mu\Phi_b)(\partial^\mu)\Phi_b\rangle := E_0,$$
$$\langle\bar{\psi}_b\psi_b\rangle := N\rho_0, \qquad \langle i\bar{\psi}_b\gamma^\mu\partial_\mu\psi_b\rangle := G_0.$$

Massive model

The massive model is given by choosing $W_1 = -m_0$ and $W_2 = \frac{1}{2} - g^2\Phi_b\Phi_b$. The terms containing F_a can be replaced via the Euler-Lagrange equation by

$$\mathcal{L}^F = \frac{g^4\bar{\psi}_a\psi_a\bar{A}A}{1-2g^2\Phi_b\Phi_b} - 2m_0g^2\bar{A}A\frac{1-2g^2\Phi_a\Phi_a}{1-4g^2\Phi_a\Phi_a} + \\ + \frac{8m_0g^6\bar{A}A(\Phi_a\Phi_a)^2}{(1-2g^2\Phi_b\Phi_b)(1-2g^2\Phi_c\Phi_c)} - \frac{m^2\Phi_a\Phi_a}{2(1-4g^2\Phi_b\Phi_b)},$$

where

$$A := \Phi_a\psi_a.$$

Only the last term will contribute in the 't Hooft limit. For ψ and Φ the Euler-Lagrange equations in the 't Hooft limit read

$$0 = (1-2Ng^2\sigma_0)i\gamma^\mu\partial_\mu\psi_a + Ng^2\rho_0\psi_a - m_0\psi_a,$$
$$0 = (1-2Ng^2\sigma_0)(\partial^\mu\partial_\mu\Phi_a) + 2g^2(E_0+G_0)\Phi_a + \\ + \frac{m_0^2}{(1-4Ng^2\sigma_0)^2}\Phi_a.$$

These are equations for free massive fermions and scalar bosons. The masses are

$$M_\psi = \frac{m_0 - Ng^2\rho_0}{1-2Ng^2\sigma_0},$$
$$M_\Phi^2 = \frac{2g^2(E_0+G_0)}{1-2Ng^2\sigma_0} + \frac{m_0^2}{(1-2Ng^2\sigma_0)(1-4Ng^2\sigma_0)^2}.$$

The expectation values can be calculated easily for the free massive case and are:

$$Ng^2\sigma_0 = \frac{Ng^2}{2\pi}\ln\frac{\Lambda}{M_\Phi},$$
$$Ng^2\rho_0 = \frac{-M_\psi Ng^2}{\pi}\ln\frac{\Lambda}{M_\psi},$$
$$g^2E_0 = M_\Phi^2 Ng^2\sigma_0,$$
$$g^2G_0 = M_\psi Ng^2\rho_0,$$

where Λ is a UV cutoff $\gg M_\Phi, M_\psi$. Solving these conditions in general is difficult because M_Φ and M_ψ have an additional logarithmic relation (given by $Ng^2\rho_0 = -2M\psi Ng^2\sigma_0 - \frac{Ng^2M_\psi}{\pi}\ln\frac{M_\Phi}{M_\psi}$), leading to a transcendental equation for $\frac{M_\Phi}{M_\psi}$ that also contains the bare coupling Ng^2.

The problematic term vanishes in the supersymmetric ansatz $M_\Phi \stackrel{!}{=} M_\psi := M$ and consequently $\rho_0 = -2M\sigma_0$. Using this ansatz in the equation for M_ψ yields

$$M = \frac{m_0 + 2MNg^2\sigma_0}{1-2Ng^2\sigma_0}. \quad \Leftrightarrow m_0 = M(1-4Ng^2\sigma_0) \tag{1}$$

Substitute m_0 in equation for M_Φ^2:

$$M^2 = \frac{2M^2Ng^2\sigma_0 - 4M^2Ng^2\sigma_0}{1-2Ng^2\sigma_0} + \frac{M^2}{1-2Ng^2\sigma_0} = M^2.$$

The gap equation 1 is, save for a factor 2 in the coupling, identical to the one of the massive Gross-Neveu model (compare [2, ch. 3.1.1]).

Massless model

The Lagrangian density of the massless model is given by $W_1 = 0$ and $W_2 = \frac{1}{2} - g^2\Phi_b\Phi_b$, the Euler-Lagrange equations are derived similarily to the massive case and read in the 't Hooft limit

$$0 = (1-2g^2N\sigma_0)i\gamma^\mu\partial_\mu\psi_a + g^2N\rho_0\psi_a,$$
$$0 = (1-2g^2N\sigma_0)\partial_\mu\partial^\mu\Phi_a + 2g^2(E_0+G_0)\Phi_a.$$

The selfconsistency conditions are

$$M_\psi = -\frac{Ng^2\rho_0}{1-2Ng^2\sigma_0},$$
$$M_\Phi^2 = \frac{2g^2(E_0+G_0)}{1-2Ng^2\sigma_0} = \frac{2g^2(NM_\Phi^2\sigma_0 + NM_\psi\rho_0)}{1-2Ng^2\sigma_0} = \\ = \frac{2Ng^2\sigma_0M_\Phi^2}{1-2Ng^2\sigma_0} - 2M_\psi^2.$$

Here the supersymmetric ansatz yields

$$0 = (1-4Ng^2\sigma_0)M$$
$$0 = (3-8Ng^2\sigma_0)M^2 \qquad \Rightarrow M = 0.$$

There is no dynamical generation of a physical mass in the case of preserved supersymmetry.
The only other selfconsistent solution where both masses are independent of the cutoff is $M_\psi = 0$, $\frac{2Ng^2}{\pi}\ln\frac{\Lambda}{M_\Phi} = 1$. This case provides a gap equation similiar to the Gross-Neveu model but the supersymmetric case is energetically favoured ($\epsilon = \frac{NM_\Phi^2}{16\pi}$ against $\epsilon_{SUSY} = 0$). In both cases the scalar density of the fermions is $\rho_0 = 0$.

CONCLUSIONS

There is a whole class of renormalizable supersymmetric invariant field theories in 1+1 dimensions with

a $O(N)$ flavour symmetry that can be considered as generalizations of the Gross-Neveu model. Examining the vacua in the 't Hooft limit for the most simple of theses generalizations yields the following results:
In the massive case there is a solution that shows the same behaviour as the respective Gross-Neveu model: The effective theory is that of N free scalar and Majorana fields with physical mass M where the relation of coupling Ng^2, UV cutoff Λ, bare mass m_0 and M is given by the gap equation $1 = \frac{m_0}{M} + \frac{2Ng^2}{\pi} \ln \frac{\Lambda}{M}$.
In the massless case the behaviour differs from the original Gross-Neveu model: No physical mass is dynamically generated in the supersymmetric case. This reproduces the result of Buchmüller and Love[3] for the NJL model in 3+1 dimensions: the supersymmetry protects the (discrete) chiral symmetry of the original Lagrangian density.

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